DUSO Mathematics League 2014 - 2015

CONTEST #3.

SOLUTIONS

3 - 1. $\boxed{4}$ Solving yields $-\frac{22}{15}x > -6 \to x < \frac{6 \cdot 15}{22} = \frac{45}{11}$. Note that $5 = \frac{55}{11} > \frac{45}{11} > \frac{44}{11} = 4$, so the greatest integer that solves the inequality is **4**.

3 - 2. $\boxed{(\mathbf{12}, \mathbf{\tfrac{2}{3}}, \mathbf{-3}) \text{ and } (\mathbf{-12}, \mathbf{-\tfrac{2}{3}}, \mathbf{3}) \textit{ [must have both]}}$ Subtracting the second equation from the first yields $xz - yz = -34$. Adding this to the third equation yields $2xz = -72 \to xz = -36$. Substituting back into the original equations yields $xy = 8$ and $yz = -2$. Now, $\frac{xz}{xy} = \frac{z}{y} = -\frac{9}{2}$, so $yz = y(-\frac{9y}{2}) = -2 \to y = \pm\frac{2}{3}$. Similarly, we see that $x = \pm 12$ and $z = \pm 3$. Making sure that the signs we choose work in all three original equations, we have $(\mathbf{12}, \mathbf{\frac{2}{3}}, \mathbf{-3})$ and $(\mathbf{-12}, \mathbf{-\frac{2}{3}}, \mathbf{3})$.

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3 - 3. $\boxed{\mathbf{1.3450}}$ Since $\sin x = \cos(90° - x)$, $\sin T = \cos R$ and $\sin R = \cos T$. The desired value is **1.3450**.

3 - 4. $\boxed{\mathbf{y = x} \text{ and } \mathbf{T_{4,4}} \textit{ (need both)}}$ Find the midpoints of the segments $\overline{BB''}$ and $\overline{OO''}$. The line through these midpoints is the line of reflection. Note that the midpoints are $B_1(2, 2)$ and $O_1(3.5, 3.5)$, so the line of reflection is $\mathbf{y = x}$. Now, the reflection maps $B(-2, 2)$ onto $B'(2, -2)$, so the translation must take $B'(2, -2)$ onto $B''(6, 2)$. That means the translation is $\mathbf{T_{4,4}}$.

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3 - 5. $\boxed{5}$ Complete the square to obtain $x^2 + 6x + 9 + y^2 - 4y + 4 = 23 + 13$, which implies $(x + 3)^2 + (y - 2)^2 = 36$. Thus, $(h, k) = (-3, 2)$ and $r = \sqrt{36} = 6$. The desired quantity is $-3 + 2 + 6 = \mathbf{5}$.

3 - 6. $\boxed{\mathbf{x \leq -5}}$ The given equation is equivalent to $|x| - |x + 5| = 5$. This can be solved using standard algebraic techniques, but consider the following. The desired values of x are those for which the distances from x to 0 and x to -5 differ by 5. This is true only for $\mathbf{x \leq -5}$.

R-1. The diagonals of square $ABCD$ intersect at E. How many distinct triangles can be formed whose vertices are at A, B, C, D, or E?

R-1Sol. $\boxed{8}$ There are four "small" triangles, each of whose areas are one-fourth the area of the square. There are also four triangles that are made of two "small" triangles, namely $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$. These are the only triangles. They total **8**.

R-2. *Let N be the number you will receive.* The number $2^N \cdot 5^4$ is written as a decimal number. Compute the sum of the digits of this number.

R-2Sol. $\boxed{7}$ Note that the product of 2 and 5 is 10. Every product of 2 and 5 will result in a terminal 0 in the decimal number. Substituting, we have $2^4 \cdot 2^4 \cdot 5^4$, or $16 \cdot 10^4$. The sum of these digits will be $1 + 6 = \mathbf{7}$.

R-3. *Let N be the number you will receive.* The point P is one-third of the way from (N, N) to $(16, -5)$. Compute the coordinates of P.

R-3Sol. $\boxed{(\mathbf{10}, \mathbf{3})}$ The coordinates of P are $\left(N + \dfrac{16 - N}{3}, N + \dfrac{-5 - N}{3}\right)$. Substituting, we obtain $(\mathbf{10}, \mathbf{3})$.

R-4. *Let P be the point you will receive.* The graph of the parabola with equation $y = ax^2 + bx + c$ passes through $(6, 3)$, $(8, 4)$, and P. Compute the product abc.

R-4Sol. $\boxed{12}$ Seeing that P is on the same horizontal line as $(6, 3)$ gives us the information that $(8, 4)$ is the vertex, so the parabola is of the form $y = a(x - 8)^2 + 4$. Substituting $(6, 3)$ yields $3 = a(4) + 4 \to a = -1/4$. Expanding the brackets yields the equation of the parabola as $y = -\frac{1}{4}x^2 + 4x - 12$. The product abc is **12**.

R-5. *Let N be the number you will receive.* A jar contains 15 balls, of which N are black and the rest are red. If three balls are chosen from the jar without replacement, compute the probability that all three are red.

R-5Sol. $\boxed{\dfrac{\mathbf{1}}{\mathbf{455}}}$ There are $15 - N = 15 - 12 = 3$ red balls. Thus, the desired probability is $\dfrac{3}{15} \cdot \dfrac{2}{14} \cdot \dfrac{1}{13} = \dfrac{\mathbf{1}}{\mathbf{455}}$.

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